**SC531 – Lecture #06**

To understand marginalisation (or summing out) slightly better, consider the following mathematical truism:

Let f(x, y, z ... ) be a function of several variables – which may be discrete, continuous or any combination of them.

Suppose we sum over – or integrate, as the case may be – over the variables y, z ... That is, all the variables except x. Then we are left with a function of only the variable x. Thus, with discrete variables:

Sall y, z ... f(x, y, z ... )

yields a function only of x, say g(x). Note that we have summed over all values of y, z ... [Essentially the same idea works with continuous variables, with *integration* in place of *summation*.]

\*\*\*

**Problem sent in by a student**:

Consider the 2-player game where the players A and B play alternately. The initial configuration is a list (1, 3, 4, 2). The play begins with player A.

At each step the player to move interchanges the position of any two elements. However a move is illegal if it leads to the same configuration as has already occurred earlier in the game.

If at any point the configuration (1,2,3,4) is reached then player A is the winner, and if (4,3,2,1) is reached then player B is the winner. It does not matter which player made the move leading to the winning configuration, only the configuration matters.

If a player to move does not have a legal move, then the game ends in a draw. For the given initial configuration, determine the outcome with the best play, with each player trying to win.

Questions:  
(1) Write down the description of a win for a player in a 2-player strategy game in the form of a predicate logic formula.

(2) Write the probability for A to win.

(3) Write the probability for B to win.

Comments:

This is a game of skill, not chance.

Total number of permutations of 4 elements = 4! = 24.

Maximum possible moves available to a player = 4C2 = 6.

Depth first search will find the best strategy for both players.

It seems that the result will be either A wins or B forces a draw, the reason being first mover advantage.

[A good discussion on search techniques is provided in Ref. #1.]

**LAPLACE’S RULE OF SUCCESSION**

Challenge or motivating question:

Evidence: We have a coin which *may be biased*. Upon tossing the coin 10 times, it shows up HEAD every time. What is the probability that it will show up HEAD on the 11th toss?

We start with a simpler problem:

Now suppose we have four boxes each of which contains 9 balls, some white and some black. The relevant composition of the boxes is:

*Box 1*: 9 white balls

*Box 2*: 4 white balls

*Box 3*: 1 white balls

*Box 4*: 5 white balls

Using our previous notation, we can say right away that the probability of drawing a white ball from a *randomly* selected box is:

P(W) = 19/36.



But how about P(B1|W), P(B2|W), P(B3|W), P(B4|W)? This is a simple application of Bayes' rule, left as an exercise.

But we can in fact say right away these four conditional probabilities are in the ratio 9:4:1:5. From that, given that they must also add up to 1, we get the four probabilities as: 9/19, 4/19, 1/19 and 5/19.

We can apply this exact approach to the original challenge problem.

Let x be the (unknown) probability that the tossed coin shows up head.

Then the probability of seeing A heads upon N tosses is:

P(A,N,x) = NCA xA (1-x)N-A [the standard *binomial expansion* term]

So we can say that P(x|A,N) must be proportional to P(A,N,x). The reasoning follows from Bayes' rule -- same as seen above, except that the variable x is continuous, while the box number was discrete.

So P(x|A,N) = k P(A,N,x) = k NCA xA (1-x)N-A.

When RHS is integrated over 0 < x < 1 and equated to 1, it yields the proportionality constant k. The final expression for P(x|A,N) is:

P(x|A,N) = (N+1) NCA xA (1-x)N-A for 0 < x < 1 [Eqn. #1]

Remember that our original question is: What is the probability of getting head on the next coin toss, that is coin toss N+1, given that there have been A heads from the N tosses so far?

Remember also that we do NOT know x, the probability of head for the coin, except of course that 0 < x < 1. Here x is fixed but unknown.

When Eqn. #1 is multiplied by x and integrated over 0 < x < 1, the result gives the expected value of x -- which is the required probability of head on toss N+1. The answer comes out as:

P(head on toss N+1|A,N) = Exp(x|A,N) = **(A+1)/(N+2)**

Note that our RATIONAL estimate of x depends on A and N.

Example: After 10 straight heads out of 10 tosses, the probability of getting a head in the 11th toss is 11/12.

An extreme example: If the sun has risen every morning for about 1012 mornings, what is the probability that it will rise tomorrow morning?

A good discussion on this topic can be found at:

<https://youtu.be/sAW8paLXGUk>

**NOTE:**

**Expected value of a random variable:**

**We know that Exp(x) = Sx x\*P(x) for discrete x, and a similar integration for a continuous random variable. Above, we have applied this same calculation to P(x|A,N).**